HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2023

STUDENT NUMBER: ______________________________

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
  Black pen is preferred
- NESA-approved calculators and drawing templates may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 12

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 The scalar product of two vectors $\underset{\sim}{u} = 3\underset{\sim}{i} - \underset{\sim}{j} - 7\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ is

(A) $6\underset{\sim}{i} - 3\underset{\sim}{j} - 7\underset{\sim}{k}$

(B) $5\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k}$

(C) –4

(D) 22

2 If $z = 3 - 4i$, then $\dfrac{1}{1-z}$ is equal to:

(A) $\dfrac{-1-2i}{10}$

(B) $\dfrac{-1+2i}{10}$

(C) $\dfrac{-1-i}{6}$

(D) $\dfrac{-1+i}{6}$

3 The equation of a sphere, centre $(2,2,3)$ that touches the $x-y$ plane only is:

(A) $(x-2)^2 + (y-2)^2 + (z-3)^2 = 9$

(B) $(x-3)^2 + (y-2)^2 + (z-2)^2 = 9$

(C) $(x-2)^2 + (y-2)^2 + (z-3)^2 = 6$

(D) $(x-2)^2 + (y-3)^2 + (z-3)^2 = 9$

4 Which of the following is equivalent to the expression $\dfrac{16x-43}{(x-3)(x^2-x-6)}$?

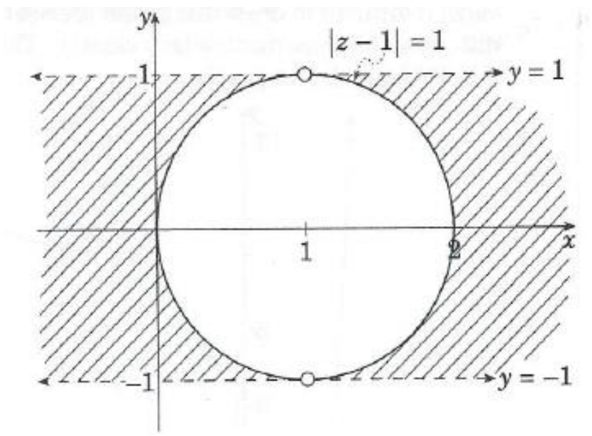
(A) $\dfrac{3}{x-3}+\dfrac{1}{x-6}-\dfrac{3}{x+1}$

(B) $\dfrac{3}{x-3}+\dfrac{1}{(x-3)^2}-\dfrac{3}{x+2}$

(C) $\dfrac{3}{x-3}+\dfrac{1}{x+3}-\dfrac{3}{x-2}$

(D) $\dfrac{3}{x-3}+\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}$

5 Which of the inequalities best represents the given region in the complex plane?



NOT TO SCALE

(A) $\{z:|z+\bar{z}|<2\}\cup\{z:|z-1|\geq 1\}$

(B) $\{z:|z+\bar{z}|<2\}\cap\{z:|z-1|\geq 1\}$

(C) $\{z:|z-\bar{z}|<2\}\cup\{z:|z-1|\geq 1\}$

(D) $\{z:|z-\bar{z}|<2\}\cap\{z:|z-1|\geq 1\}$

6 Which expression is equivalent to $\int (\ln x)^2 dx$

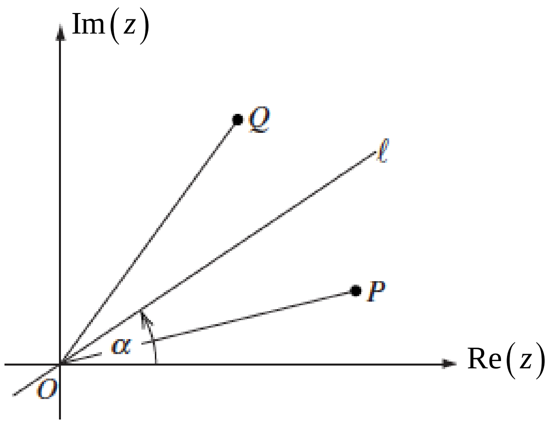
(A) $x(\ln x)^2 - 2\int \ln x dx$

(B) $(\ln x)^2 - 2\int \ln x dx$

(C) $x(\ln x)^2 - 2\int x \ln x dx$

(D) $2\int \ln x dx$

7 l is a line in the Argand diagram that passes through the origin and makes an angle α with the positive real axis, where $0 \le \alpha \le \frac{\pi}{2}$.



The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line l to produce the point Q, which represents the complex number z_2.
Hence $|z_1| = |z_2|$.

The value of $\arg(z_1) + \arg(z_2)$ is

(A) α

(B) 2α

(C) 3α

(D) none of the above.

8 What is the converse of the following statement?

"If both m and n are divisible by d, then $m-n$ is divisible by d."

(A) If $m-n$ is divisible by d, then both m and n are divisible by d.

(B) If $m-n$ is not divisible by d, then both m and n are not divisible by d.

(C) If $m-n$ is not divisible by d, then neither m nor n is not divisible by d.

(D) If both m and n are not divisible by d, then $m-n$ is not divisible by d.

9 Consider the statement:

'$\forall x \in R,$ if $f(x)$ is even, then $f''(x)$ is also even'

Which of the following is the negation to the statement?

(A) $\exists x \in R,$ if $f(x)$ is not even, then $f''(x)$ is also not even

(B) $\exists x \in R,$ if $f''(x)$ is not even, then $f(x)$ is also not even

(C) $\forall x \in R,$ if $f(x)$ is not even, then $f''(x)$ is also not even

(D) $\forall x \in R,$ if $f''(x)$ is not even, then $f(x)$ is also not even

10 Which best describes the following parametric equations?

$$x = -5\cos t$$
$$y = 5\sin t$$
$$z = t$$

(A) A spiral around the y axis moving in an anticlockwise direction.

(B) A spiral around the y axis moving in a clockwise direction.

(C) A spiral around the z axis moving in an anticlockwise direction.

(D) A spiral around the z axis moving in a clockwise direction.

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) (i) Given that $a > 0, b > 0$, prove that $a + b \geq 2\sqrt{ab}$ **2**

(ii) Hence show $\sec^2 x \geq 2\tan x$ **1**

(b) (i) Express $z = \sqrt{2} - i\sqrt{2}$ in modulus-argument form. **2**

(ii) Hence evaluate z^{24} **1**

(c) The point P represents the complex number $-2 + 4i$. The point Q is in the first quadrant of the Argand diagram such that $\angle OQP = \frac{\pi}{2}$ and $|OQ| = |QP|$. **2**

Find the complex number represented by the point Q.

(d) Find the primitive function of each of the following:

(i) $\int \frac{e^x}{1 + e^{2x}}\ dx$ **2**

(ii) $\int \sin^2 x.\cos^3 x\ dx$ **2**

(iii) $\int \frac{2x + 1}{x^2 + 2x + 2}\ dx$ **3**

End of the Question 11.

Question 12 (15 marks) Start a new writing booklet

(a) (i) Show that one of the zeros of the equation $2z^3 - 5z^2 + 6z - 2 = 0$ is $1 - i$. **1**

(ii) Hence, or otherwise, find the other zeros over the complex field C. **2**

(b) Consider the statement 'If $5^x = 2$, $x \in R$, then x is irrational.'

(i) Find the converse of the above statement. **1**

(ii) Prove that the above statement is true. **2**

(c) Find $\int x \sin^{-1} x \; dx$ **4**

(d) (i) Point P divides the line segment AB internally in the ratio of m:n, prove that **2**

$$\underset{\sim}{p} = \frac{m\underset{\sim}{b} + n\underset{\sim}{a}}{m+n}$$

where $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(ii) Point P is the midpoint of AB, show that $\underset{\sim}{p} = \dfrac{\underset{\sim}{a} + \underset{\sim}{b}}{2}$. **1**

(iii) Points P, Q and R are midpoints of the sides of ΔACB, where $\overrightarrow{OC} = \underset{\sim}{c}$. **2**
Show that $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$.

End of Question 12

Question 13 (15 marks) Start a new writing booklet

(a) The line l_1 passes through the points $A(4,-3,-3)$ and $B(5,2,2)$.

(i) Find the vector equation of line l_1. **2**

(ii) The equation of line l_2 is given by $\frac{x-1}{k}=\frac{y}{-1}=\frac{z+3}{1}$ **3**

Find the value of k for which l_1 and l_2 intersect.

(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x-2\sin x}$ **4**

(c) The spheres $x^2+y^2+z^2=9$ and $x^2+(y-4)^2+z^2=16$ intersect. Find

(i) the value of y when they intersect. **1**

(ii) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius. **2**

(d) Use mathematical induction to prove for all integers $n \geq 2$, **3**

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}<2-\frac{1}{n}$$

You may assume $\frac{1}{(k+1)^2}-\frac{1}{k}+\frac{1}{k+1}<0$ is true.

End of Question 13

Question 14 (15 marks) Start a new writing booklet

(a) Use a counter-example to show that the statement 'a quadrilateral is formed by joining any four points in a plane. **1**

(b) 'If $\forall x \in R,\ x^2 - 6x + 5$ is even, then x is odd.'

(i) Write down a contrapositive statement. **1**

(ii) Use contraposition to prove the original conjecture. **2**

(c) (i) Sketch the set of points representing the complex number z such that **2**

$$\arg(z+2) = \frac{\pi}{4} + \arg(z-2i)$$

(ii) Find the radius and centre. **2**

(d) (i) If ω is a complex root of $z^3 - 1 = 0$ with the smallest positive argument, show that ω^2 is also a cube root of unity. **2**

(ii) Show that $1 + \omega + \omega^2 = 0$. **1**

(iii) Find all the roots in the form $e^{i\theta}$ and indicate these roots in an Argand diagram. **2**

(iv) Show that $\left(1+\omega^2\right)^5 = -\omega^2$ **2**

End of Question 14

Question 15 (15 marks) Start a new writing booklet

(a) Given $a^2 + b^2 \geq 2ab$ (Do NOT prove this)

(i) Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$ **1**

(ii) Deduce that if $x + y + z = 1$, then $xy + yz + zx \leq \frac{1}{3}$ **2**

(b) (i) Given $z = \cos\theta + i\sin\theta$, prove $z^n + \frac{1}{z^n} = 2\cos n\theta$ **2**

(ii) Hence by considering the expansion $\left(z + \frac{1}{z}\right)^4$ show that **3**

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(iii) Hence find the primitive of $\int \cos^4\theta \, d\theta$ **1**

(c) Find the shortest distance from the point $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ to the line $r = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ **3**

(d) A sequence is defined by $a_1 = 2$, $a_2 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for all integers $n \geq 3$ **3**

Prove by mathematical induction that $a_n = 2^{n-1} + 1$ for integers $n \geq 1$

End of Question 15

Question 16 (15 marks) Start a new writing booklet

(a) Find the integral $\int_0^{\frac{\pi}{4}} \sec^3 x \ dx$. **3**

(b) Lines l_1 and l_2 are given below, relative to a fixed point O.

$$l_1 : \ r = \left(11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}\right) + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - p\underset{\sim}{k}\right)$$

$$l_2 : \ r = \left(-5\underset{\sim}{i} + 11\underset{\sim}{j} + q\underset{\sim}{k}\right) + \mu\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

where λ and μ are scalar parameters.

(i) Given that lines l_1 and l_2 are perpendicular, find the value of p. **2**

(ii) Given that lines l_1 and l_2 intersect perpendicularly, find the value of q. **3**

(iii) Hence find the point of intersection of lines l_1 and l_2. **1**

(c) (i) Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \ dx$ **2**

(ii) Show that $0 \le x^n \le x^2$, for $n \ge 2$ in the above integral. **1**

(iii) Hence show that for $n \ge 2$, $\frac{1}{2} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \le \frac{\pi}{6}$. **3**

End of Examination

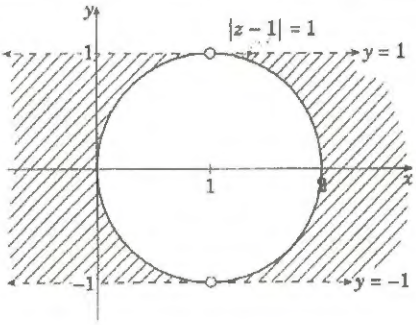
HGHS Ext 2 2023 TRIAL HSC SOLUTIONS

Marking Guidelines:

Multiple-choice Answer key

1	C
2	A
3	A
4	B
5	D
6	A
7	B
8	A
9	A
10	D

1	$\underset{\sim}{u} = 3\underset{\sim}{i} - \underset{\sim}{j} - 7\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ $\underset{\sim}{u} \bullet \underset{\sim}{v} = 3(2) + (-1)(3) + (-7)(1)$ $\underset{\sim}{u} \bullet \underset{\sim}{v} = -4$ **(C)**	Correct answer:
2.	$\frac{1}{1-z} = \frac{1}{1-(3-4i)}$ $= \frac{1}{-2+4i} \times \frac{-2-4i}{-2-4i}$ $= \frac{-2-4i}{20}$ $= \frac{-1-2i}{10}$ **(A)**	Correct answer:
3.	Centre (2, 2, 3) and touches the $x-y$ plane, i.e. $z = 0$ i.e. radius is 3 units $(x-2)^2 + (y-2)^2 + (z-3)^2 = 9$ **(A)**	Correct answer:
4.	$\frac{16x-43}{(x-3)(x^2-x-6)} = \frac{16x-43}{(x-3)(x-3)(x+2)}$ $= \frac{16x-43}{(x-3)^2(x+2)}$ $= \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+2}$ $= \frac{A(x-3)(x+2) + B(x+2) + C(x-3)^2}{(x-3)^2(x+2)}$ $16x-43 = A(x-3)(x+2) + B(x+2) + C(x-3)^2$ When $x = 3$, $5 = 5B$ $\therefore B = 1$ $x = -2$, $-75 = 25C$ $\therefore C = -3$ $x = 0$, $-43 = -6A + 2 - 27$ $-18 = -6A$ $\therefore A = 3$ $\therefore \frac{3}{x-3} + \frac{1}{(x-3)^2} - \frac{3}{x+2}$ **(B)**	Correct answer:

<table>
<tr><td>5.</td><td>


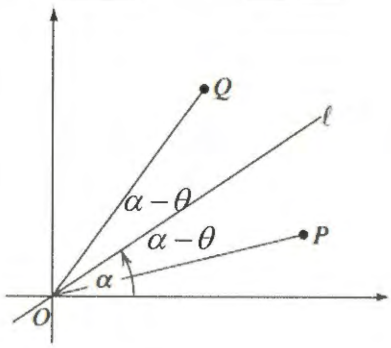
$|z-\bar{z}|\ge 1$

$|z-\bar{z}|=|2i|<2$

$|i|<1$

$\{z:|z-\bar{z}|<2\}\cap\{z:|z-1|\ge 1\}$ **(D)**
</td><td>Correct answer:</td></tr>
<tr><td>6.</td><td>
$\int(\ln x)^2\ dx = uv-\int u'v\ dx$ $\quad u=(\ln x)^2 \quad v'=1$

$u'=\frac{2\ln x}{x} \quad v=x$

$=x(\ln x)^2-2\int \ln x\ dx$ **(A)**
</td><td>Correct answer:</td></tr>
<tr><td>7.</td><td>


Let $\arg(z_1)=\theta$

Angle between z_1 and line $l=\alpha-\theta$

$\therefore \arg(z_1)+\arg(z_2)=\theta+(\alpha+\alpha-\theta)$

$=2\alpha$ **(B)**
</td><td>Correct answer:</td></tr>
<tr><td>8.</td><td>
\P = both m and n are divisible by d
Q = $m-n$ is divisible by d
Converse: If $Q\Rightarrow P$ **(A)**
</td><td>Correct answer:</td></tr>
</table>

9.

Key words	Negation
$\forall x \in R$, is even	$\exists x \in R$, is not even

$\exists x \in R$, if $f(x)$ is not even, then $f''(x)$ is also not even **(A)**

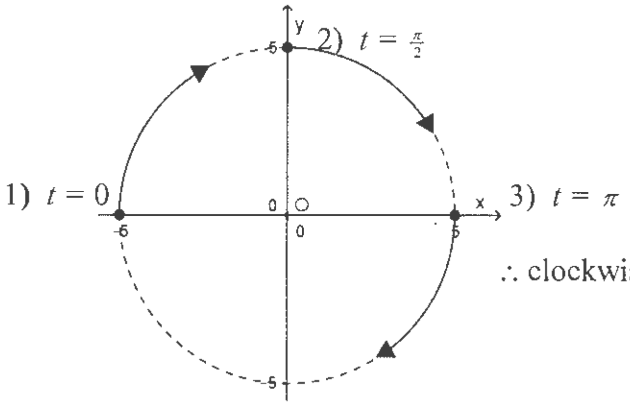
Correct answer:

10.

$x = -5\cos t$
$y = 5\sin t$
$z = t$

i.e. spiral up z – axis as t increases.

Method 1: viewing from the $x - y$ plane



$\therefore$ clockwise

(D)

Method 2: Table of values

	$t = 0$	$t = \dfrac{\pi}{2}$	$t = \pi$
$x = -5\cos t$	$x = -5$	$x = 0$	$x = 5$
$y = 5\sin t$	$y = 0$	$y = 5$	$y = 0$
$z = t$	$z = 0$	$z = \dfrac{\pi}{2}$	$z = \pi$

$\therefore$ turning clockwise around the z axis.

Correct answer:

QUESTION 11 (a)

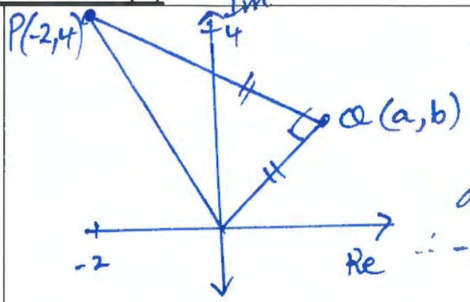
(i)	$(\sqrt{a}-\sqrt{b})^2 \geq 0$ $a-2\sqrt{ab}+b \geq 0$ $a+b \geq 2\sqrt{ab}$ $a>0,\ b>0$ equality when $a=b$ (2)	Mostly well done. All but 1 got it correct
(ii).	$\sec^2 x = \tan^2 x + 1$ $\geq 2\sqrt{\tan^2 x . 1}$ (using (i)) $= 2\lvert \tan x \rvert$ $= 2\tan x$ as $\tan^2 x \geq 0$ (1)	Mostly well done. All but 2 got it correct

QUESTION 11 (b)

(i)	$z = \sqrt{2} - i\sqrt{2}$ $\lvert z \rvert = \sqrt{2+2}$ $= 2$ $\arg z = \tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)$ $= \tan^{-1}(-1)$ $= -\frac{\pi}{4}$ $\therefore z = 2\,\text{cis}\left(-\frac{\pi}{4}\right)$ $= 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ 2	Well done. 1 incorrect

(ii)	$z^{24} = 2^{24}\,\text{cis}\left(-\frac{\pi}{4}\right)^{24}$ $= 2^{24}\,\text{cis}(-6\pi)$ $= 2^{24}(\cos 6\pi - i\sin 6\pi)$ $= 2^{24}(1-0)$ $= 2^{24}$ (16777216) 1	Well done. 2 incorrect

QUESTION 11 (c)

	 let $\vec{OQ} = a+bi$ $\vec{QP} = \vec{OP} - \vec{OQ}$ $= (-2+4i)-(a+bi)$ also $\vec{QO} = i\,\vec{QP}$ $\therefore -(a+bi) = i[(-2+4i)-(a+bi)]$ $= (b-4)+(-2-a)i$ $\therefore$ equating real & imag $\left.\begin{array}{l} -a = b-4 \\ -b = -2-a \end{array}\right\}$ gives $a=1,\ b=3$ $\therefore \vec{OQ} = 1+3i$	A lot of students had $OP \perp OQ$ when it is $PQ \perp OQ$ Not many got this correct (2)

QUESTION 11 (d)

(i)	$\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du$ let $u = e^x$, $du = e^x\,dx$ $= \tan^{-1} u + c$ $= \tan^{-1} e^x + c$ (2)	generally well done All but 2 got it correct

QUESTION 11 (d)

(ii)	$\displaystyle\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx$ $\displaystyle= \int \sin^2 x(1-\sin^2 x)\cos x\,dx$ $\displaystyle= \int \cos x \sin^2 x - \cos x \sin^4 x\,dx$ $\displaystyle= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$ (2)	generally well done All but 3 got it correct

QUESTION 11 (d)

(iii)	$\displaystyle\int \frac{2x+1}{x^2+2x+2} = \int \frac{2x+2-1}{x^2+2x+2}\,dx$ $\displaystyle= \int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{1}{x^2+2x+2}\,dx$ $\displaystyle= \ln(x^2+2x+2) - \int \frac{1}{(x+1)^2+1}\,dx$ $= \ln(x^2+2x+2) - \tan^{-1}(x+1) + c$ [3]	generally well done all but 3 got it correct

QUESTION 12 (a)

(i) Let $P(z)=2z^3-5z^2+6z-2$

$$P(1-i)=2(1-i)^3-5(1-i)^2+6(1-i)-2$$

$$P(1-i)=2\left(1-3i+3i^2-i^3\right)-5\left(1-2i+i^2\right)+6-6i-2$$

$$=2(1-3i-3+i)-5(1-2i-1)+4-6i$$

$$=2(-2-2i)-5(-2i)+4-6i$$

$$=-4-4i+10i+4-6i$$

$$=0$$

(1 mark)

Hence $1-i$ is a root of $P(z)=2z^3-5z^2+6z-2$.

DNK: |||

Expansion error $(-i)^3$: 1

Poor working out: ~~||||~~ ~~||||~~ |

No conclusion: 1

(ii) Since $1-i$ is a root, coefficients of $P(z)$ are real, then its conjugate $1+i$ is also a root

i.e. $(1-i)+(1+i)=2$

$$(1-i)(1+i)=2$$

$$\therefore [z-(1-i)][z-(1+i)]=z^2-2z+2 \text{ is a factor}$$

$$\begin{array}{r}
2z-1 \\
z^2-2z+2\,\big)\,2z^3-5z^2+6z-2 \\
-\left(2z^3-4z^2+4z\right) \\
\hline
-z^2+2z-2 \\
-\left(-z^2+2z-2\right) \\
\hline
0
\end{array}$$

Hence the other roots are $1+i$ and $\frac{1}{2}+0i$ over $\mathbb{C}$.

ϕ 1

AE: 1

Had $\alpha+\beta+\gamma=-\frac{d}{a}$ ||

Did not explain why $1-i$ is

Had $1-i$ as the other roots: ||

Did not answer the question for the other roots: ~~||||~~ ||||

Factorised $P(z)$ as answer: ||

DNK: 1

QUESTION 12 (b)

(i)	'If $5^x = 2,\ x \in R,$ then x is irrational.' where $P = '5^x = 2,\ x \in R'$ and $Q = x$ is irrational The converse statement (i.e. $Q \Rightarrow P$) is 'If x is irrational, $x \in R,$ then $5^x = 2.$'	Wrong: 111 forgot '$x \in R$' : 11
(ii)	Using contradiction, 'If $5^x = 2,\ x \in R,$ then x is rational.' If $5^x = 2,\ x \in R,$ where x is rational i.e. $x = \frac{p}{q}$ where $p,\ q \in Z^+$ and $\frac{p}{q}$ in simplest form. $5^{\frac{p}{q}} = 2$ $\left(5^{\frac{p}{q}}\right)^q = (2)^q$ $5^p = 2^q$ Since LHS $= 5^p$ $= 5 \times 5 \times 5 \times ... \times 5$ where there is no factor that is divisible by 2. or $5 \times 5 \times 5 \times ...$ is odd. and RHS $= 2^q$ $= 2 \times 2 \times 2 \times ... \times 2$ where there is no factor that is divisible by 5. or $2 \times 2 \times 2 \times ...$ is even. Hence the proposal of $x = \frac{p}{q}$ is rational, is a contradiction therefore the original statement is true. OR	Did not state contradiction: ~~HHHH~~ Did not explain ~~factor~~ fully: 1111 ϕ 1 DNK: 1 Proven by contrapositive then said statement is a contradiction: 1 Conceptual problem with depiction: 11 Proven by contradiction and concluded by converse 1 Got contrapositive statement wrong: 1

By contrapositive "If x is rational, then $5^x \neq 2$, $x \in \mathbb{R}$".

No conclusion!

QUESTION 12 (c)

Let $\displaystyle\int x\sin^{-1}x\ dx$

$$= uv - \int u'v\,dx \quad \text{where } u = \sin^{-1}x \text{ and } v' = x$$

$$u' = \frac{1}{\sqrt{1-x^2}} \longrightarrow v = \frac{x^2}{2}$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx \qquad \text{Let } x = \sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{\sin^2\theta}{|\cos\theta|}\cos\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\sin^2\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{1}{2}(1-\cos2\theta)\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\int 1-\cos2\theta\,d\theta$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\theta + \frac{\sin2\theta}{2}\right] + C$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\sin^{-1}x + x\sqrt{1-x^2}\right] + C$$

DNK what to do with $\int\frac{x^2}{\sqrt{1-x^2}}\,dx$: IIII

Had $\cos\theta = \frac{1}{\sqrt{1-x^2}}$: I

ϕ : I

DNK: $\int\sqrt{1-x^2}\,dx$: IIII

accepted: $\frac{\sin(2\sin^{-1}x)}{8}$

DNK: $v' = \sin^{-1}x$ I

DNK : II

careless transcription : I

Had $\int u'v'\,dx$: I

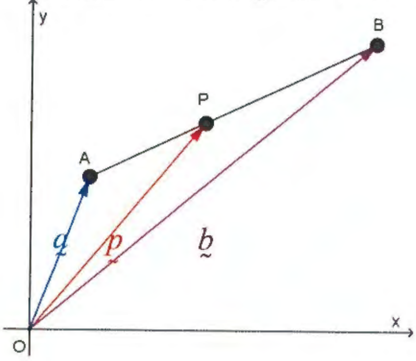
QUESTION 12 (c) continues

	$=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{2}\int\cos^2\theta\,d\theta$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\int(1+\cos 2\theta)\,d\theta$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\left[\theta+\frac{\sin 2\theta}{2}\right]+C$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\left[\sin^{-1}x+\frac{2\sin\theta\cos\theta}{2}\right]+C$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\left[\sin^{-1}x+\sin\theta\cos\theta\right]+C$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\left[\sin^{-1}x+x\sqrt{1-x^2}\right]+C$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{2}\sin^{-1}x+\frac{1}{4}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}+C$ $=\frac{x^2\sin^{-1}x}{2}-\frac{1}{4}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}+C$	

QUESTION 12 (d)

(i)

. Point P divides the line segment AB internally in the ratio of $m{:}n$.



Point P divides the line segment AB internally the ratio of $m{:}n$.

$$\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB} \quad \text{where } \overrightarrow{AP} : \overrightarrow{PB} = m : n$$

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{m}{n}$$

$$\therefore n\overrightarrow{AP} = m\overrightarrow{PB}$$

$$n\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = m\left(\overrightarrow{OB} - \overrightarrow{OP}\right)$$

$$n\left(\underset{\sim}{p} - \underset{\sim}{a}\right) = m\left(\underset{\sim}{b} - \underset{\sim}{p}\right)$$

$$n\underset{\sim}{p} - n\underset{\sim}{a} = m\underset{\sim}{b} - m\underset{\sim}{p}$$

$$n\underset{\sim}{p} + m\underset{\sim}{p} = m\underset{\sim}{b} + n\underset{\sim}{a}$$

$$(n+m)\underset{\sim}{p} = m\underset{\sim}{b} + n\underset{\sim}{a}$$

$$\therefore \underset{\sim}{p} = \frac{m\underset{\sim}{b} + n\underset{\sim}{a}}{m+n}$$

Marker's comments:

DNKa: |||

Error in algebraic technique: |

ϕ: ||

Had $\overrightarrow{AP} = \frac{m}{n}(\underset{\sim}{b} - \underset{\sim}{a})$: |

mostly did well using vectors properly.

Using division of interval but did not know how to relate to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as well as $\overrightarrow{OP}$: ||||

(ii)

Point P is the midpoint of AB, i.e. $m : n = 1 : 1$

$$\therefore m = n = 1$$

$$\therefore \underset{\sim}{p} = \frac{(1)\underset{\sim}{b} + (1)\underset{\sim}{a}}{(1)+(1)}$$

$$\therefore \underset{\sim}{p} = \frac{\underset{\sim}{b} + \underset{\sim}{a}}{2}$$

Marker's comments:

Inadequate explanation: |||

(iii)

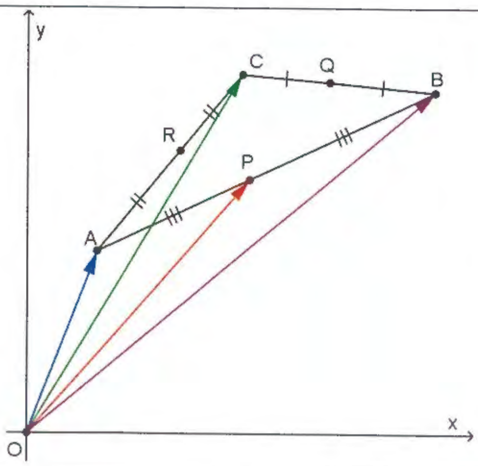
$$\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r}$$

$$= \frac{\underset{\sim}{a} + \underset{\sim}{b}}{2} + \frac{\underset{\sim}{a} + \underset{\sim}{c}}{2} + \frac{\underset{\sim}{b} + \underset{\sim}{c}}{2}$$

$$= \frac{2\underset{\sim}{a} + 2\underset{\sim}{b} + 2\underset{\sim}{c}}{2}$$

$$= \frac{2\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)}{2}$$

$$\therefore \underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$$



Marker's comments:

Mostly did well.

QUESTION 13 (a)

(i)	$A(4,-3,-3) \quad B(5,2,2)$ $\vec{AB} = \vec{OB} - \vec{OA}$ $= \begin{pmatrix}5\\2\\2\end{pmatrix} - \begin{pmatrix}4\\-3\\-3\end{pmatrix}$ $= \begin{pmatrix}1\\5\\5\end{pmatrix}$ 2 $\therefore$ Eqn of l_1 (say $\underset{\sim}{r}$): $\underset{\sim}{r} = \begin{pmatrix}4\\-3\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\5\end{pmatrix}$ * Note: there are others such as $\underset{\sim}{r} = \begin{pmatrix}5\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\5\end{pmatrix}$	Mostly well done
(ii)	$l_2 : \frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} = \mu$ (say) For l_1: $x = 4+\lambda$ — ① For l_2: $x = 1+k\mu$ — ② $y = -3+5\lambda$ — ③ $\quad y = -\mu$ — ④ $z = -3+5\lambda$ — ⑤ $\quad z = -3+\mu$ — ⑥ from ⑤ and ⑥ $-3+5\lambda = -3+\mu$ 1 $\mu = 5\lambda$ ⊛ from ③ and ④ $-3+5\lambda = -\mu$ sub ⊛ $\quad -3+5\lambda = -5\lambda$	Mostly well done except a few algebraic errors. Some need to learn method
	$10\lambda = 3$ 1 $\lambda = \frac{3}{10}$ then $\mu = \frac{3}{2}$ } sub into ① & ② $4+\lambda = 1+k\mu$ $4+\frac{3}{10} = 1+\frac{3}{2}k \Rightarrow k = \frac{11}{5}$ 1	3

QUESTION 13 (b)

$\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x - 2\sin x}$ $= \int_0^1 \frac{1}{3-\frac{(1-t^2)}{1+t^2} - \frac{4t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$ $= 2\int_0^1 \frac{dt}{3+3t^2-1+t^2-4t}$ $= 2\int_0^1 \frac{dt}{4t^2-4t+2}$ 1 $= 2\int_0^1 \frac{dt}{(2t-1)^2+1}$ let $t = \tan\frac{x}{2}$ $\frac{dt}{dx} = \frac{\sec^2\frac{x}{2}}{2}$ $= \frac{1+t^2}{2}$ $x = \frac{\pi}{2} \quad t = 1$ 1 $x = 0 \quad t = 0$ $= 2\left[\frac{\tan^{-1}(2t-1)}{2}\right]_0^1$ 1 $= [\tan^{-1}(2\times1-1) - \tan^{-1}(2\times0-1)]$ 1 $= \tan^{-1}(1) - \tan^{-1}(-1)$ $= \frac{\pi}{4} - -\frac{\pi}{4} = \frac{\pi}{2}$	Generally well done to begin with. • Watch the bounds • constant divider was a problem for some

QUESTION 13 (c)

<table>
<tr><td>(i)</td><td>

$x^2+y^2+z^2=9$ —— ①

$x^2+(y-4)^2+z^2=16$ —— ②

② − ①

$(y-4)^2-y^2=7$

$y^2-8y+16-y^2=7$

$8y=9$

$y=\frac{9}{8}$ —— ③ $\therefore$ they intersect on the plane $y=\frac{9}{8}$

</td><td>Really well done.</td></tr>
<tr><td>(ii)</td><td>

sub ③ into ①

$x^2+\left(\frac{9}{8}\right)^2+z^2=9$

$x^2+\frac{81}{64}+z^2=9$

$x^2+z^2=\frac{495}{64}$

$\therefore$ Circle centre $\left(0,\frac{9}{8},0\right)$ radius $\frac{3\sqrt{55}}{8}$ u

</td><td>Mostly well done except for a couple of algebraic errors.</td></tr>
</table>

QUESTION 13 (d)

RTP $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}<2-\frac{1}{n}$, $n\geq 2$ ① Prove for $n=2$ $LHS=\frac{1}{1^2}+\frac{1}{2^2}=\frac{5}{4}$ $RHS=2-\frac{1}{2}=3/2$ as $5/4<3/2$ then true for $n=2$ 1 ② Assume true for $n=k$ i.e. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{k^2}<2-\frac{1}{k}$ ③ Prove for $n=k+1$ i.e $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$ 1 $LHS=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$ $<2-\frac{1}{k}+\frac{1}{(k+1)^2}$ using assumption $=2+\frac{1}{(k+1)^2}-\frac{1}{k}+\frac{1}{k+1}-\frac{1}{k+1}$ $<2-\frac{1}{k+1}$ as $\frac{1}{(k+1)^2}-\frac{1}{k}+\frac{1}{k+1}<0$ given 1	Generally well done • Some need to watch their step 1 as they were a little careless • Step 3 mostly well done • Step 4 should be better even though marks are not generally awarded for it – however they can be deducted.

$< RHS$

④ ∴ if result is true for $n=k$, then it is true for $n=k+1$.

Since it is true for $n=2$, then it is true for $n=3$.

If it is true for $n=3$, then it is true for $n=4$ and so on.

∴ it is true for all integer $n\geq 2$.

QUESTION 14 (a)

'A quadrilateral is formed by joining any four points in a plane.'
This statement is false when the four points are collinear, then a line is formed.
OR This statement is false when the three points are collinear, then with the fourth point, a triangle is formed.

QUESTION 14 (b)

(i) 'If $\forall x \in R,\ x^2 - 6x + 5$ is even, then x is odd.' i.e. $P \Rightarrow Q$

Given	Negation
$\forall x \in R$	$\exists x \in R$
is even	is not even
is odd	is not odd

$P =$ If $\forall x \in R,\ x^2 - 6x + 5$ is even

$Q =$ x is odd

Contrapositive: $\neg Q \Rightarrow \neg P$

$\therefore$ Contrapositive statement: If, $\exists x \in R$, x is not odd, then $x^2 - 6x + 5$ is not even.

OR If, $\exists x \in R$, x is even, then $x^2 - 6x + 5$ is odd.

4 used $\exists x \in R$

(ii) Let $x = 2n,\ n \in \mathbb{Z}$ (definition of even)

$$
\begin{aligned}
x^2 - 6x + 5 &= (2n)^2 - 6(2n) + 5 \\
&= 4n^2 - 12n + 5 \\
&= 4n^2 - 12n + 4 + 1 \\
&= 2\left(2n^2 - 6n + 2\right) + 1 \\
&= 2m + 1,\ m \in \mathbb{Z} \text{ i.e. definition of odd.}
\end{aligned}
$$

~~Since true by contraposition,~~

Hence the original conjecture is proven by contraposition.

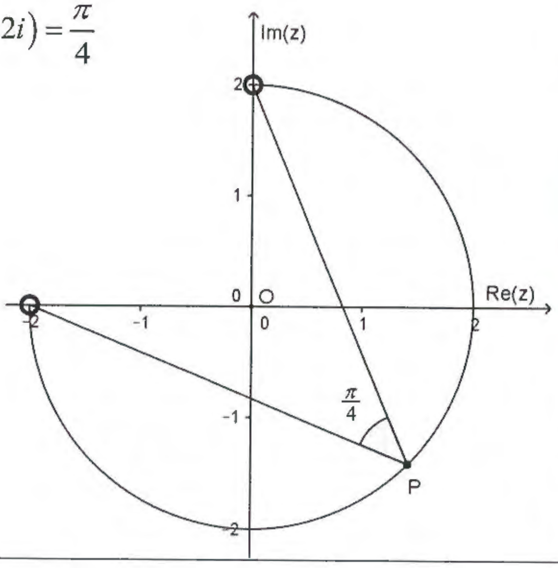
~~Many~~ A few did not conclude properly to show that they understand how contraposition is related to original statement.

QUESTION 14 (c)

(i)

$$\arg(z+2)=\frac{\pi}{4}+\arg(z-2i)$$

$$\arg(z+2)-\arg(z-2i)=\frac{\pi}{4}$$



Many student did not realised if $\angle APB=\frac{\pi}{4}$ then $\angle AOB=\frac{\pi}{2}$.

A few forgot open circles.

(ii) Centre is at (0, 0)

Radius is 2 units.

QUESTION 14 (d)

(i)

$z^3-1=0$

$z^3=1$ where $z=\cos\theta+i\sin\theta$

$z^3=\cos 3\theta+i\sin 3\theta$ by de Moirve's Theorem

$\cos 3\theta=1$

$3\theta=2k\pi+0,\ k=0,\ 1,\ 2$

$\therefore \theta=\frac{2k\pi}{3}$

When $k=0,\ \theta=0,\ \therefore z_1=1$

$k=1,\ \theta=\frac{2\pi}{3},\ \therefore z_2=cis\frac{2\pi}{3}=\omega$

$k=2,\ \theta=\frac{4\pi}{3},\ \therefore z_3=cis\frac{4\pi}{3}=cis\,2\left(\frac{2\pi}{3}\right)=\omega^2$

Hence ω^2 is a root of $z^3-1=0$.

Many approached from different method. Some students still did not know how to do.

(ii)

ω is a complex root, hence $\omega^3=1$

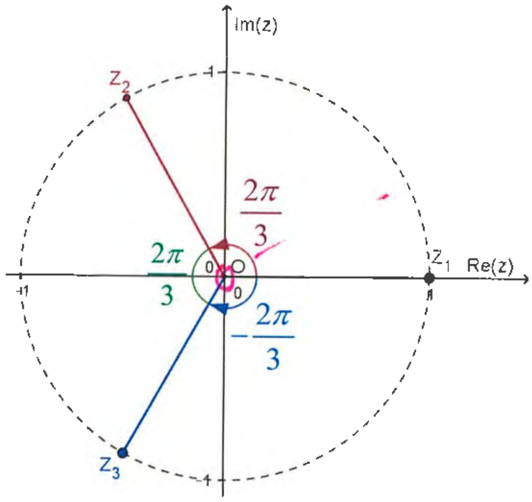
$$\omega^3-1=0$$

$$(\omega-1)\left(1+\omega+\omega^2\right)=0 \quad \text{but } \omega\neq 1$$

$$\therefore 1+\omega+\omega^2=0$$

Mostly done well.

QUESTION 14 (d)

(iii)	From (i) When $k=0,\ \theta=0,\ \therefore z_1=e^{i0}$ $k=1,\ \theta=\frac{2\pi}{3},\ \therefore z_2=e^{\frac{2\pi}{3}i}$ $k=2,\ \theta=\frac{4\pi}{3},\ \therefore z_3=e^{-\frac{2\pi}{3}i}$ 	Open circle at origin
(iv)	From (ii) $1+\omega+\omega^2=0$ i.e. $1+\omega=-\omega^2$ or $\omega+\omega^2=-1$ or $1+\omega^2=-\omega$ $\therefore\left(1+\omega^2\right)^5=(-\omega)^5$ $=-\omega^5$ $=-\omega^3\times\omega^2$ since $\omega^3=1$ $=-(1)\times\omega^2$ $=-\omega^2$ (as required)	Expanded as binomial and got lost!

QUESTION 15 (a)

(i)	given $a^2+b^2 \geq 2ab$ then $x^2+y^2 \geq 2xy$ — ① $x^2+z^2 \geq 2xz$ — ② $y^2+z^2 \geq 2yz$ — ③ so ①+②+③ $2x^2+2y^2+2z^2 \geq 2xy+2xz+2yz$ 1 so $x^2+y^2+z^2 \geq xy+xz+yz$ #	Generally well done
(ii)	given $x+y+z=1$ $(x+y+z)^2 = 1 = x^2+y^2+z^2+2(xy+xz+yz)$ $\therefore x^2+y^2+z^2 = 1-2(xy+xz+yz)$ $\therefore 1-2(xy+xz+yz) \geq xy+xz+yz$ (from (i)) $1 \geq 3(xy+xz+yz)$ $\therefore xy+xz+yz \leq \frac{1}{3}$ 2	Generally well done

QUESTION 15 (b)

(i)	given $z = \cos\theta + i\sin\theta$ then $z^n = \cos n\theta + i\sin n\theta$ 1 $z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ $= \cos n\theta - i\sin n\theta$ $\therefore z^n + z^{-n} = 2\cos n\theta$ 2	generally well done
(ii)	$\left(z+\frac{1}{z}\right)^4 = z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$ $= \left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$ 1 $= 2\cos 4\theta + 4(2\cos 2\theta)+6$ $= 2\cos 4\theta + 8\cos 2\theta + 6$ 1 also $\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4$ $= 16\cos^4\theta$ 3	generally well done
~~(iii)~~ (iii)	$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ 1 $\cos^4\theta = \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$ $\therefore \int \cos^4\theta \, d\theta = \int \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8} \, d\theta$ $= \frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8} + C$ 1	generally well done

QUESTION 15 (c)

Let $A = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and P be a pt. on $r = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Poorly done question. About 5 students did this correctly

Say $P = \begin{bmatrix} 1+2\lambda \\ -\lambda \\ 1+\lambda \end{bmatrix}$

Now $\vec{AP} = \begin{bmatrix} 1+2\lambda \\ -\lambda \\ 1+\lambda \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -2+2\lambda \\ -1-\lambda \\ -1+\lambda \end{bmatrix}$ 1

$|\vec{AP}|$ is minimum when $\vec{AP} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$ (right angles)

$\therefore \begin{bmatrix} -2+2\lambda \\ -1-\lambda \\ -1+\lambda \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 0 \Rightarrow -4+4\lambda+1+\lambda-1+\lambda = 0$

$6\lambda - 4 = 0$

$\lambda = 2/3$ 1

then $P = \begin{bmatrix} 1+2\cdot 2/3 \\ -2/3 \\ 1-2/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -2/3 \\ 5/3 \end{bmatrix}$

QUESTION 15 (d)

$\therefore$ min distance is

$\left| \begin{bmatrix} 7/3 \\ -2/3 \\ 5/3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right| = \left| \begin{bmatrix} -2/3 \\ -5/3 \\ -1/3 \end{bmatrix} \right|$

$= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(-\frac{5}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}$

$= \frac{\sqrt{30}}{3}$ units 1

(d) $a_1 = 2,\ a_2 = 3$ & $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$

Prove $a_n = 2^{n-1} + 1$

for $n=1$ $a_1 = 2^{1-1}+1 = 1+1 = 2$

$n=2$ $a_2 = 2^{2-1}+1 = 2+1 = 3$

$\therefore a_n = 2^{n-1}+1$ is true for $n = 1, 2$ 1

A lot of students did not do $n=1$ and $n=2$. The value of a_n is dependent on 2 values proceeding it

Assume true for
$n=1$ $a_1 = 2$
$n=2$ $a_2 = 3$
$n=3$ $a_3 = 2^2+1 = 5$
$\vdots$
$n=k-1$ $a_{k-1} = 2^{k-2}+1$
$n=k$ $a_k = 2^{k-1}+1$

Most did this correctly. A few students did not show the sequence in the assumption.

Prove for $n = k+1$, $a_{k+1} = 2^{(k+1)-1}+1$ 1

$= 2^k + 1$

(over page).

Now $a_{k+1} = 3a_k - 2a_{k-1}$ by recurrance relation

$$= 3(2^{k-1}+1) - 2(2^{k-2}+1) \text{ by assumption}$$

$$= 3.2^{k-1} + 3 - 2.2^{k-2} - 2$$

$$= 3.2^{k-1} - 2.2^{k-2} + 1$$

$$= 3.2^{k-1} - 2^{k-1} + 1$$

$$= 2.2^{k-1} + 1$$

$$= 2^k + 1$$

Most did this step well.

1

which is true for $n = k+1$ if true for $n = 1 \leq n \leq k$, $k \geq 1$ where k is an integer

Since a_k is true for $n=1, n=2$ it is true for $n=3$

Since a_k is true for $n=2, n=3$ it is true for $n=4$

and so on

$\therefore a_n = 2^{n-1} + 1$ is true for all $n \geq 1$.

This could be a lot better for nearly everyone.

QUESTION 16 (a)

	$\int_0^{\frac{\pi}{4}} \sec^3 x \ dx = \int_0^{\frac{\pi}{4}} \sec^2 x \sec x \ dx$ $= [uv]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} u'v \ dx$ where $u = \sec x$ $u' = \sec x \tan x$ $v' = \sec^2 x$ $v = \tan x$ $= [\sec x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec x \tan^2 x \ dx$ $= \left[\sec\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right] - \int_0^{\frac{\pi}{4}} \sec x\left(\sec^2 x - 1\right) \ dx$ $= \left[\sqrt{2}(1)\right] - \int_0^{\frac{\pi}{4}} \sec^3 x - \sec x \ dx$ Let $I = \int_0^{\frac{\pi}{4}} \sec^3 x \ dx$ $I = \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 x \ dx + \int_0^{\frac{\pi}{4}} \sec x \ dx$ $2I = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec x \ dx$ $2I = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec x \times \frac{\sec x + \tan x}{\sec x + \tan x} \ dx$ $2I = \sqrt{2} + \int_0^{\frac{\pi}{4}} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \ dx$ $2I = \sqrt{2} + \left[\ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{\pi}{4}}$ $2I = \sqrt{2} + \left[\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(1+0)\right]$ $2I = \sqrt{2} + \left[\ln\left(\sqrt{2}+1\right) - \ln\left(\sec 0 + \tan 0\right)\right]$ $2I = \sqrt{2} + \left[\ln\left(\sqrt{2}+1\right) - \ln 1\right]$ $2I = \sqrt{2} + \ln\left(\sqrt{2}+1\right)$ $\therefore I = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)$	Mostly done well.

QUESTION 16 (b)

(i)

$l_1 : r = \left(11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}\right) + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - p\underset{\sim}{k}\right)$

$$= \begin{bmatrix} 11 \\ 2 \\ 17 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ -p \end{bmatrix} \text{ with direction vector } \begin{bmatrix} -2 \\ 1 \\ -p \end{bmatrix};$$

$l_2 : r = \left(-5\underset{\sim}{i} + 11\underset{\sim}{j} + q\underset{\sim}{k}\right) + \mu\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

$$= \begin{bmatrix} -5 \\ 11 \\ q \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} \text{ with direction vector } \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix};$$

For $l_1 \perp l_2$, $\begin{bmatrix} -2 \\ 1 \\ -p \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = 0$

$$(-2)(-3) + (1)(2) \cancel{+} - 2p = 0$$

$$6 + 2 \cancel{+} - 2p = 0$$

$$8 \cancel{+} - 2p = 0$$

$$-2p = -8$$

$$\therefore p = \cancel{-}4$$

Many did not show that $\begin{bmatrix} -2 \\ 1 \\ -p \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = 0$.

(ii)

$$r_1 = \begin{bmatrix} 11 \\ 2 \\ 17 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} \text{ and } r_2 = \begin{bmatrix} -5 \\ 11 \\ q \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

Equating the like components:

$11 - 2\lambda = -5 - 3\mu$ — ①

$2 + \lambda = 11 + 2\mu$ — ②

$17 - 4\lambda = q + 2\mu$ — ③

From ② $\lambda = 9 + 2\mu$ — ②'

Sub ②' into ① $11 - 2(9 + 2\mu) = -5 - 3\mu$

$$11 - 18 - 4\mu = -5 - 3\mu$$

$$\therefore \mu = -2 \text{ — ④}$$

Mostly well done.

QUESTION 16 (b)(ii) continues

(ii)	Sub ④ into ②' $\lambda = 9 + 2(-2)$ $\therefore \lambda = 5$ — ⑤ Sub ④ and ⑤ into ③ $17 - 4(5) = q + 2(-2)$ $-3 = q - 4$ $\therefore q = 1$	
(iii)	$r_1 = \begin{bmatrix} 11 \\ 2 \\ 17 \end{bmatrix} + 5\begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$ $r_1 = \begin{bmatrix} 11-10 \\ 2+5 \\ 17-20 \end{bmatrix}$ $r_1 = \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}$ $\therefore$ The point of intersection is (1, 7, -3) Checking: $r_2 = \begin{bmatrix} -5 \\ 11 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$ $r_2 = \begin{bmatrix} -5+6 \\ 11-4 \\ 1-4 \end{bmatrix}$ $r_2 = \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}$	Did not answer as a "point". Many did not answer ~~as~~ as a point.

$\cancel{26 + 6\lambda} =$ 169
$\cancel{26}$ 12
$6\lambda = 26 - \frac{7}{13}$ $\frac{338}{7}$
$= \frac{331}{13}$

QUESTION 16 (c)

(i)

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx = \left[\sin^{-1} x\right]_0^{\frac{1}{2}}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

All correct.

(ii)

For $n \geq 2$, since $0 \leq x \leq \frac{1}{2}$,

$$0 \leq x^n \leq \left(\frac{1}{2}\right)^n \quad \text{and} \quad 0 \leq x^2 \leq \left(\frac{1}{2}\right)^2$$

$$0 \leq \left(\frac{1}{2}\right)^n \leq \left(\frac{1}{2}\right)^2$$

$$\therefore 0 \leq x^n \leq x^2$$

Many missed the point $0 \leq x \leq \frac{1}{2}$

(iii)

For $n \geq 2$ and $0 \leq x \leq \frac{1}{2}$, $\quad 0 \leq x^n \leq x^2$

$$1 \geq 1 - x^n \geq 1 - x^2$$

$$1 \geq \sqrt{1-x^n} \geq \sqrt{1-x^2}$$

$$1 \leq \frac{1}{\sqrt{1-x^n}} \leq \frac{1}{\sqrt{1-x^2}}$$

$$\int_0^{\frac{1}{2}} 1\, dx \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}\, dx \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\left[x\right]_0^{\frac{1}{2}} \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}\, dx \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\frac{1}{2} - 0 \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}\, dx \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\therefore \frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}\, dx \leq \frac{\pi}{6}$$

Many did not start from $0 \leq x^n \leq x^2$ hence did not explain the inequality